A SAWI DECOMPOSITION-BASED METHOD FOR THE SOLUTION OF FRACTIONAL BRATU-TYPE DIFFERENTIAL EQUATIONS

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O. O. OLUBANWO[1], A. S. AJANI[1], M. A. AYODELE[1], F. H. ALAMU-AWONIRAN[2], A. I. ADEJUMO[3], O. E. SOTONWA[4]

[1]Department of Mathematical Sciences, Olabisi Onabanjo University, Ago-Iwoye, Ogun State, Nigeria
[2]School of Science and Technology, Basic Science Department, Babcock University,Ilisan, Ogun State, Nigeria
[3]Mathematics Programme, College of Agriculture, Engineering and Science, Bowen University, Iwo, Osun State, Nigeria
[4]Department of Statistics, Gateway ICT Polytechnic, Saapade, Nigeria
[*]Corresponding Author: olubanwo.oludapo@oouagoiwoye.edu.ng

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Abstract

The paper employs the Sawi Decomposition Method to solve fractional Bratu type differential equations, obtaining an analytical approximation of the nonlinear Fractional Bratu-type differential equation. The approximate solution obtained converges to a series of infinitely computable terms. The way in which the exact and approximate analytical solutions behave for various variables of ∝ are visually plotted. This stated approach (Sawi Decomposition Method) was employed to solve three special cases of fractional Bratu type differential equations to obtain the approximate results for each example and the results obtained were compared to those of other approaches already in use. It was verified that the suggested method's outcome completely matched the outcomes of other approaches. The Sawi Decomposition Method basically requires lesser computational efforts compared with other existing method. MATLAB-generated diagrams demonstrate the effectiveness of the method in capturing the behaviours of fractional nonlinear problems. This research advances numerical techniques for solving nonlinear PDEs and contributes to the field of fractional differential equations.

Keywords: Sawi Transform, Adomian Polynomials, Adomian Decomposition Method, Iteration, Integral transform.

Introduction

Several engineering domains have developed models using fractional calculus. Calculus is being used in a wider range of fields, such as electrochemistry, networks, engineering, and viscoelasticity (Podlubny, 1999). Systems that fall under the categories of signal processing, biology, chemistry, physics, material science, and dynamic control theory (Kilbas et al.2006, Kirchner et al., 2000). In scientific and engineering fields, fractional differential equations, both linear and nonlinear, have proven effective in simulating phenomena. Scholars like Podlubny (1999) and Kilbas et al. (2006) have provided in-depth explanations of the calculus notion and its applications. Fascinatingly, fractional derivatives are defined throughout the whole time domain in physical processes, while integer order derivatives are solely connected to local properties at certain moments.
Saad et al. (2018) investigated a number of fractional calculus-related topics. An analysis of Saad's academic writings provides insight into the evolution of fractional derivatives. The Burger's, Korteweg-de Vries-Burger's, and Korteweg-de Vries-Burger's equations are examples of equations with Liouville-Caputo fractional space derivatives that have been estimated in the real world using the spectral collocation technique based on shifted Chebyshev polynomials (Saad et al. 2018).

Additionally, a new model combining fractional-order quadratic autocatalysis and linear inhibition was proposed, and the solution was approximated using power law, exponential law, and the Mittag-Leffler kernel (Saad et al. 2020). Numerous integral transformations have been widely applied in engineering and physics. These transformations have been the major focus of several researchers on their theory and application. They have been utilized sequentially to solve simultaneous linear equations and differential equations. Several integral transforms are widely recognized, such as the Elzaki (2011), Fourier (1978), Kamal (Kamal and Sedeeg, 2016), Aboodh (2013), Sumudu (1993), Mohand (Mahgoub, 2017), Sawi (Mahgoub, 2019).

Mohand Mahgoub presented Sawi, a novel integral transform (Mahgoub, 2019). It has shown to be useful in mathematical modeling and analysis when applied to solve ordinary differential equations in control engineering applications. Many characteristics of the Sawi transform have been studied and documented in academic publications (Halim and Zakaria 2023), emphasizing its aptitude and efficiency in managing intricate mathematical procedures and issues that arise in engineering-related fields.

Over the set of functions of exponential order, the Sawi transform is defined (Mahgoub, 2019)

$$A = \left\{ f(t) : \exists\, M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{k_i}},\ if\ t \in (-1)^i X[0, \infty) \right\} \qquad 1$$

The constant M for a particular function in the set A must be a finite value, although k_1 and k_2 might be finite or infinite.

The Sawi transform which is usually represented by $S(.)$ is expressed as an integral equations

$$S[f(t)] = L(u) = \frac{1}{u^2} \int_0^\infty f(t)\, e^{\frac{t}{u}} dt,\ t \geq 0,\ k_1 \leq u \leq k_2 \qquad 2$$

Many scholars concentrate especially on approximations of fractional order functional equation solutions when there isn't an exact solution for differential equations of fractional

order. Differential equations of fractional order may now be solved using a variety of numerical methods that were first developed to solve differential equations of integer order. Along with other approaches, these strategies include the VIM (Odibat and Momani 2006), ADM (Sadeghinia and Kumar 2015) and HPM (Kumar et al. 2013).
Here, we will concentrate on using the Sawi Decomposition Method to solve Bratu-type fractional differential equations.

Materials and Methods

Examine the following fractional Bratu differential equation (Manjare and Dinde, 2020)

$$D^{\alpha}y(t) + L(y) + N(t-\tau) = g(t), \tau \in R, v < r \quad 3$$

With initial condition $n-1 < \alpha \le n$

$$v^k(0) = v_0^k$$

Hence L refers to the linear bounded operator and N refers to the nonlinear bounded operator, $g(t)$ refers to the given continuous function $D^{\alpha}y(t)$.

The Sawi Decomposition Method is used in this study to approximate the analytical solution of equation (2) in the case when there are no linear bounded operator's L and g(t) in the initial conditions. Considering that the Bratu-type differential lacks a linear time and function of t. Therefore, only Bratu-type differential equations of the form will be addressed in this study.

$$D^{\alpha}y(t) + L(t-\tau) = 0 \ldots, r \in L, t < \tau, n-1 < \alpha \le n \quad 4$$

Subject to

$$v^k(0) = v_0^k$$

Thus, L refers to the nonlinear bounded operator and refers to fractional order derivative term. Applying Sawi Transform on equation on equation (4)

$$S\{D^{\alpha}y(t)\} + S\{L(t-\tau)\} = 0$$

$$\frac{1}{v^{\alpha}}S\{y(t)\} - \sum_{i=0}^{m-1}\left(\frac{1}{v}\right)^{\alpha-(i-1)} g(i)(0) + S\{L(t-\tau)\} = 0 \quad 5$$

Where $B = \sum_{i=0}^{m-1} g(i)(0)$

$$\frac{S\{y(x)\}}{v^{\alpha}} - \frac{B}{v^{\alpha-(i-1)}} + S\{L(t-\tau)\} = 0$$

$$= S\{y(x)\} = BV^{i-1} - V^{\alpha}S\{L(t-\tau)\} \quad 6$$

The Sawi decomposition method defined solution $y(t)$ by the series

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \quad 7$$

The nonlinear operator can be decomposed as

$$L(t-\tau) = \sum_{n=0}^{\infty} C_n \quad 8$$

Where C_n as Admonian polynomials of $y_0, y_1, y_2, \ldots y_n$ that are given by:

$$C_n = \frac{1}{n!}\frac{d^n}{dX^n}[L(\sum_{n=0}^{\infty} X^n y_n)]_{\lambda=0}$$

$$n = 0,1,2,\ldots$$

The Adomian polynomials may be obtained from

$$C_0 = L(y_0)$$

$$C_1 = y_1 L'(y_0)$$

$$C_2 = y_2 L'(y_0) + \frac{1}{2!}y_1^2 L''(y_0)$$

$$C_3 = y_3 L'(y_0) + y_1 y_2 L''(y_0) + \frac{1}{3!}y_1^3 L'''(y_0) \quad 9$$

Substituting (7) and (8) into (6), we have

$$\int\{\sum_{n=0}^{\infty} y_n\}BV^{i-1} - V^{\alpha}S\{\sum_{n=0}^{\infty} C_n\} \quad 10$$

Equating both sides of equation (10)

$$n = 0, S\{y_0\} = BV^{i-1} \quad 11$$

$$n = 1, S\{y_1\} = -V^{\alpha}S\{C_0\} \quad 12$$

$$n = 2, S\{y_2\} = -V^{\alpha}S\{C_1\} \quad 13$$

The recursive relation is written as

$$S\{y_n\} = -V^{\alpha}S\{C_{n-1}\}, n \ge 1 \quad 14$$

Taking inverse Sawi Transform of equation (11), (12), (13) and (14) yields

$$y_0 = G(t) \quad 15$$

$$y_n = -S^{-1}\{U^{\alpha}S\{C_{n-1}\}\}, n \ge 1 \quad 16$$

Consequently, G(t) is a function that arises from the starting circumstances specified and the source term.

Results

Example 1:

Examine the following differential equation of Bratu-type with fractional order of the kind (Manjare and Dinde, 2020)

$$D^{\alpha}y(t) - 2e^{y(t)} = 0, 1 < \alpha \le 2, 0 < x < 1 \quad 17$$

with

$$y(0) = 0, y'(0) = 0$$

The closed form solution is $y(x) = -2lncosx$ for $\propto = 2$.In order to derive the approximate solution. Taking the Sawi transform of equation (17)

$$S\{D^{\alpha}y(t)\} - 2S\{e^{y(t)}\} = 0 \quad 18$$

$$\{D^{\alpha}y(t)\} = 2S\{e^{y(t)}\}$$

$$\frac{S\{y(t)\}}{V^{\alpha}} - \frac{y(0)}{V^{\alpha+1}} - \frac{y'(0)}{V^{\alpha}} = 2S\{e^{y(t)}\} \quad 19$$

Substituting the initial condition in equation (18)

$$\frac{S\{y(t)\}}{V^{\alpha}} = 2S\{e^{y(t)}\}$$

$$S\{y(t)\} = 2VS\{e^{y(t)}\}$$

Taking the Sawi Inverse transform of equation (17)

$$\frac{y(t)}{y_0(t)} = S^{-1}\left\{2V^{\alpha}S\{e^{y(t)}\}\right\} \quad 20$$

$$y_0(t) = S^{-1}(0) = 0 \quad 21$$

$$y_{n+1}(t) = 2S^{-1}\{V^{\alpha}S\{C_n\}\} \quad 22$$

Here, the nonlinear term is $L = e^{y(t)}$

From equation (9)

$$C_0 = e^{y_0(t)})$$

$$C_1 = y_1(t)e^{y_0(t)}$$

$$C_2 = y_2 L'(t) + \frac{1}{2!}y_1^2(t)L''(y_0(t)) \quad 23$$

From equation (22)

$$y_{n+1}(t) = 2S^{-1}\{V^{\alpha}S\{C_n\}\}$$

When $n = 0, y_1(t) = 2S^{-1}\{V^{\alpha}S\{C_0\}\}$ 24

Substituting equation (23) into (24) yields

$$y_1(t) = 2S^{-1}\left\{V^{\alpha}S\{e^{y_0(t)}\}\right\}$$

$$= 2S^{-1}\{V^{\alpha}S\{e^0\}\}$$

$$= 2S^{-1}\{V^{\alpha}S\{1\}\}$$

$$= 2S^{-1}\left\{V^{\alpha}\frac{1}{V}\right\}$$

$$S\{1\} = \frac{1}{V}$$

$$= 2S^{-1}\{V^{\alpha-1}\}$$

$$= \frac{2t^{\alpha}}{\Gamma(\alpha+1)} \quad 25$$

Putting $n = 1$ in equation (22)

$$y_2(t) = 2S^{-1}\{V^{\alpha}S\{C_1\}\} \quad 26$$

Substituting equation (23) into (26) yields

$$y_2(t) = 2S^{-1}\left\{V^{\alpha}S\{y_1(t).e^{y_0(t)}\}\right\}$$

$$= 2S^{-1}\left\{V^{\alpha}S\left\{\frac{2t^{\alpha}}{\Gamma(\alpha+1)}.e^0\right\}\right\}$$

$$= 2S^{-1}\left\{V^{\alpha}S\left\{\frac{2t^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}$$

$$= 2S^{-1}\left\{V^{\alpha}\frac{2}{\Gamma(\alpha+1)}S\left\{\frac{t^{\alpha}}{\Gamma}\right\}\right\}$$

$$= 2S^{-1}\left\{V^{\alpha}\frac{2}{\Gamma(\alpha+1)}V^{\alpha-1}\Gamma(\alpha+1)\right\}$$

$$= 4S^{-1}\{V^{2\alpha-1}\}$$

$$y_2(t) = 4\frac{t^{2\alpha}}{\Gamma}(2\alpha+1) \quad 27$$

Putting $n = 2$ into equation (22)

$y_3(t) = 2S^{-1}\{V^{\alpha}S\{C_2\}\}$

Substituting equation (23) equation (27) yields

$y_3(t) = 2S^{-1}\left\{V^{\alpha}S\left\{y_2L'(y_0) + \frac{1}{2!}y_1^2L''(y_0)\right\}\right\}$

$$= 2S^{-1}\left\{V^{\alpha}S\left\{\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}e^{y_0(t)} + \frac{1}{2!}\left(\frac{2t^{\alpha}}{\Gamma(\alpha+1)}\right)^2 e^{y_0(t)}\right\}\right\}$$
$$= 2S^{-1}\left\{V^{\alpha}S\left\{\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}e^{0} + \frac{1}{2!}\frac{4t^{2\alpha}}{(\Gamma(\alpha+1))^2}e^{0}\right\}\right\}$$
$$= 8S^{-1}\left\{V^{\alpha}S\left\{\left(\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{2\Gamma(2\alpha+1)(\Gamma(\alpha+1))^2}\right)t^{2\alpha}\right\}\right\}$$
$$= 8S^{-1}\left\{V^{\alpha}\left(\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{2\Gamma(2\alpha+1)(\Gamma(\alpha+1))^2}\right)S(t^{2\alpha})\right\}$$
$$= 8S^{-1}\left\{V^{\alpha}\left(\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{2\Gamma(2\alpha+1)(\Gamma(\alpha+1))^2}\right)V^{2\alpha-1}.\Gamma(2\alpha+1)\right\}$$
$$y_3 = 8\left[\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{2\Gamma(2\alpha+1)(\Gamma(\alpha+1))^2}\right]S^{-1}\{V^{3\alpha-1}.\Gamma(2\alpha+1)\}$$
$$= 8\left[\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{2\Gamma(2\alpha+1)(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.\Gamma(2\alpha+1)\right]$$
$$y_3 = 4\left[\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\right]t^{3\alpha} \quad 28$$

The solution is obtained as

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots$$
$$y(t) = 0 + \frac{2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{4t^{2\alpha}}{\Gamma(2\alpha+1)} + 4\left[\frac{2[\Gamma(\alpha+1)]^2+\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\right]t^{3\alpha} \quad 29$$

Equation (29) is the solution of equation (17) which is exactly the same as the solution obtained in Manjare, and Dinde, 2020. In a special case when $\alpha = 2$, equation (29) yields

$$y(t) = t^2 + \frac{t^4}{6} + \frac{2}{45}t^6 + \cdots \quad 30$$

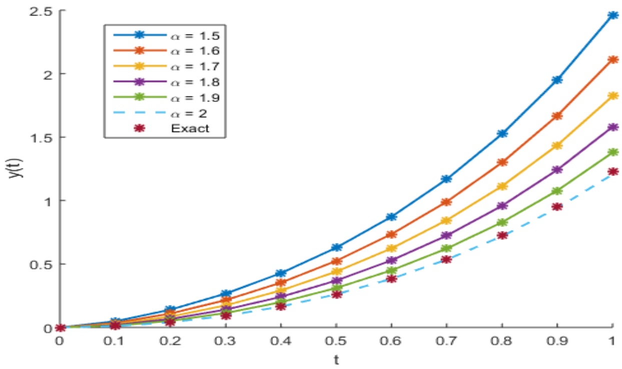

Figure 1. Behaviour of the equation when $\alpha = 1.5, \alpha = 1.6, \alpha = 1.7, \alpha = 1.8, \alpha = 1.8, \alpha = 1.9, \alpha = 2.0$, and exact solution

Table 1.Behaviour of the equation when $\alpha = 1.5, \alpha = 1.6, \alpha = 1.7, \alpha = 1.8, \alpha = 1.8, \alpha = 1.9, \alpha = 2.0$, and exact solution

T	alpha =1.5	alpha-1.6	alpha=1.7	alpha=1.8	alpha=1.9	alpha=2	Exact
0	0	0	0	0	0	0	0
0.1	0.048252	0.035469	0.025992	0.018983	0.013814	0.010017	0.010017
0.2	0.140107	0.109603	0.085629	0.066764	0.051925	0.040269	0.04027
0.3	0.266499	0.215352	0.174092	0.140651	0.113479	0.091378	0.091383
0.4	0.427966	0.352826	0.291451	0.240935	0.199134	0.164426	0.164458
0.5	0.628046	0.524673	0.439766	0.369346	0.310492	0.261024	0.261168
0.6	0.872284	0.735334	0.622609	0.528819	0.450057	0.383414	0.38393
0.7	1.167935	0.990864	0.844984	0.7235	0.621311	0.534592	0.536172
0.8	1.523895	1.298917	1.113375	0.958839	0.828844	0.718461	0.722781
0.9	1.950704	1.668806	1.43585	1.241739	1.078523	0.940017	0.950885
1	2.460577	2.111597	1.822203	1.580725	1.377694	1.205556	1.231253

Example 2
Examine the following differential equation of Bratu-type with fractional order of the kind (Manjare, and Dinde, 2020)
$D^{\alpha}y(t) - e^{2y(t)} = 0, 1 < \alpha \leq 2, 0 < t < 1$ 31
S.t $y(0) = 0,\ \ y'(0) = 0$ 32
The closed form solution is $y(x) = lnsecx$ for $\alpha = 2$
Taking Sawi Transform of equation (31)
$S\{D^{\alpha}y(t)\} - S\{e^{2y(t)}\} = 0$
$S\{D^{\alpha}y(t)\} = S\{e^{2y(t)}\}$ 33
$\frac{S\{y(t)\}}{V^{\alpha}} - \frac{y(0)}{V^{\alpha+1}} - \frac{y'(0)}{V^{\alpha}} = S\{e^{2y(t)}\}$ 34
Substituting the initial condition in equation 34
$\frac{S\{y(t)\}}{V^{\alpha}} - \frac{0}{V^{\alpha+1}} - \frac{0}{V^{\alpha}} = S\{e^{2y(t)}\}$
$\frac{S\{y(t)\}}{V^{\alpha}} = S\{e^{2y(t)}\}$
$S\{y(t)\} = V^{\alpha}S\{e^{2y(t)}\}$ 35
Taking the Sawi Inverse Transform of Equation (35)
$S^{-1}\{S\{y(t)\}\} = S^{-1}\left\{V^{\alpha}S\{e^{2y(t)}\}\right\}$
$y(t) = S^{-1}\left\{V^{\alpha}S\{e^{2y(t)}\}\right\}$
$y_0(t) = S^{-1}\{0\} = 0$
$y_{n+1}(t) = S^{-1}\{V^{\alpha}S\{C_n\}\}$ 36
Here, the nonlinear term is $L = e^{2y(t)}$
Recall from equation (9), we have
$C_0 = e^{2y_0(t)})$
$C_1 = 2y_1(t)e^{2y_0(t)}$
$C_2 = 2y_2(t)e^{2y_0(t)} + 2y_1^2(t)e^{2y_0(t)}$ 37
Putting $n = 0$ in equation (36)
$y_1(t) = S^{-1}\{V^{\alpha}S\{C_0\}\}$
$= S^{-1}\left\{V^{\alpha}S\{e^{2y_0(t)}\}\right\}$ 38
$= S^{-1}\left\{V^{\alpha}S\{e^{2(0)}\}\right\}$
$= S^{-1}\{V^{\alpha}S\{1\}\}$
$= S^{-1}\{V^{\alpha}V^{-1}\}$
$= S^{-1}\{V^{\alpha-1}\}$
$y_1(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$ 39
Putting $n = 1$ in equation (36)
$y_2(t) = S^{-1}\{V^{\alpha}S\{C_1\}\}$ 40
Substituting equation (37) into (40)
$y_2(t) = S^{-1}\left\{V^{\alpha}S\{2y(t)e^{2y_0(t)}\}\right\}$
$y_2(t) = S^{-1}\left\{V^{\alpha}S\left\{\frac{2t^{\alpha}}{\Gamma(\alpha+1)}e^{0}\right\}\right\}$
$= S^{-1}\left\{V^{\alpha}S\left\{\frac{2t^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}$
$= S^{-1}\left\{V^{\alpha}\frac{2V^{\alpha-1}}{\Gamma(\alpha+1)}.\Gamma(\alpha+1)\right\}$
$y_2(t) = S^{-1}\{2V^{2\alpha-1}\}$
$= \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}$ 41
Next putting $n = 2$ in equation (7) yields
$y_3(t) = S^{-1}\{V^{\alpha}S\{C_2\}\}$
$= S^{-1}\left\{V^{\alpha}S\{2y_2(t)e^{2y_0(t)} + 2y_1^2(t)e^{2y_0(t)}\}\right\}$
$= S^{-1}\left\{V^{\alpha}S\left\{\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}e^{2(0)} + 2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^2 e^{2(0)}\right\}\right\}$
$= S^{-1}\left\{V^{\alpha}S\left\{\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(\alpha+1)^2}e^{2(0)}\right\}\right\}$
$y_3(t) = S^{-1}\left\{V^{\alpha}S\left\{\frac{2\left[\left(\Gamma(\alpha+1)\right)^2+\Gamma(2\alpha+1)\right]}{\Gamma(2\alpha+1)\left(\left(\Gamma(\alpha+1)\right)^2\right.}t^{2\alpha}\right\}\right\}$
$= S^{-1}\left\{V^{\alpha}\frac{2\left[\left(\Gamma(\alpha+1)\right)^2+\Gamma(2\alpha+1)\right]}{\Gamma(2\alpha+1)\left(\left(\Gamma(\alpha+1)\right)^2\right.}.S\{t^{2\alpha}\}\right\}$
$= S^{-1}\left\{V^{\alpha}\frac{2\left[\left(\Gamma(\alpha+1)\right)^2+\Gamma(2\alpha+1)\right]}{\Gamma(2\alpha+1)\left(\left(\Gamma(\alpha+1)\right)^2\right.}V^{2\alpha-1}.\Gamma(2\alpha+1)\right\}$

$$= S^{-1}\left\{\frac{2\left[(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{\Gamma(2\alpha+1)((\Gamma(\alpha+1))^2}V^{3\alpha-1}.\Gamma(2\alpha+1)\right\}$$
$$= \frac{2\left[(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{\Gamma(2\alpha+1)((\Gamma(\alpha+1))^2}\Gamma(2\alpha+1)S^{-1}\{V^{3\alpha-1}\}$$
$$= \frac{2\left[(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{((\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$y_3(t) = \frac{2\left[(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{\Gamma(3\alpha+1)((\Gamma(\alpha+1))^2}t^{3\alpha} \quad 42$$

The solution obtained is written in series form as

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots$$
$$y(t) = 0 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2\left[(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{\Gamma(3\alpha+1)((\Gamma(\alpha+1))^2}t^{3\alpha} \quad 43$$

Equation (43) is the solution of equation (31) which is the same the result obtained in (Manjare and Dinde, 2020). In a special case when $\alpha = 2$, equation(43) becomes

$$y(t) = \frac{t^2}{2!} + \frac{2t^4}{4!} + \frac{18}{6!}t^6 + \cdots \quad 44$$

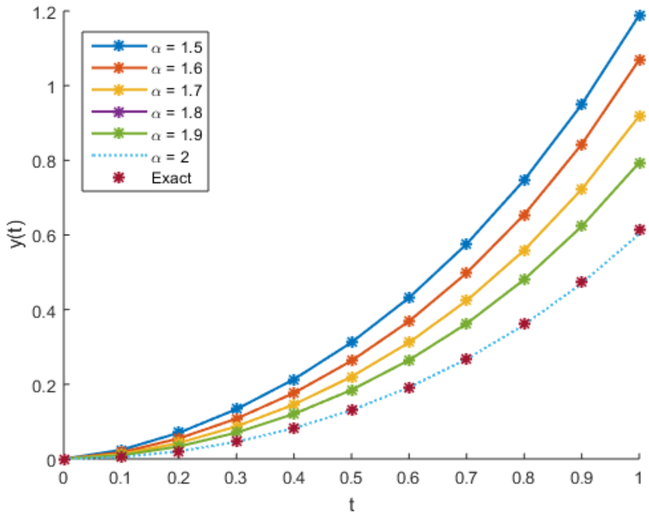

Figure 2: Behaviour of the equation when α=1.5, α=1.6, α=1.7, α=1.8, α=1.8, α=1.9, α=2.0, and exact solution

Table 2. Behaviour of the equation when $\alpha = 1.5, \alpha = 1.6, \alpha = 1.7, \alpha = 1.8, \alpha = 1.8, \alpha = 1.9, \alpha = 2.0$, and exact solution

t	alpha =1.5	alpha-1.6	alpha=1.7	alpha=1.8	alpha=1.9	alpha=2	Exact Solution
0	0	0	0	0	0	0	0
0.1	0.024125	0.017735	0.012996	0.009491	0.006907	0.005008	0.005008
0.2	0.070024	0.054808	0.042817	0.033383	0.025963	0.020135	0.020135
0.3	0.133065	0.107718	0.087062	0.070331	0.056741	0.045689	0.045692
0.4	0.213309	0.17658	0.145794	0.120492	0.099573	0.082213	0.082229
0.5	0.312184	0.262823	0.220095	0.184754	0.155269	0.130512	0.130584
0.6	0.431963	0.368833	0.311842	0.264627	0.225094	0.191707	0.191965
0.7	0.575605	0.497875	0.423672	0.362249	0.310812	0.267296	0.268086
0.8	0.746698	0.654096	0.559019	0.480446	0.414757	0.359231	0.361391
0.9	0.949443	0.842567	0.722176	0.622809	0.539917	0.470009	0.475442
1	1.188663	1.069335	0.918377	0.793788	0.690041	0.602778	0.615626

Example 3

Examine the following differential equation of Bratu-type with fractional order of the kind (Manjare and Dinde, 2020)

$$D^{2\alpha}y(t) - \pi^2 e^{y(t)} = 0, 0 < \alpha \leq 2, 0 < x < 1 \quad 45$$

with

$$y(0) = 0,\ \ y^\alpha(0) = \pi \quad 46$$

The closed form solution is

$y(t) = -\ln(1 - sin\pi t)$ for $\alpha = 2$
Taking Sawi Transform of equation (45)
$S\{D^\alpha y(t)\} - S\{\pi^2 e^{y(t)}\} = 0$
$S\{D^\alpha y(t)\} = S\{\pi^2 e^{y(t)}\}$ 47
$\frac{S\{y(t)\}}{V^{2\alpha}} - \frac{y(0)}{V^{2\alpha-0}} - \frac{y^\alpha(0)}{V^{2\alpha-\alpha}} = \pi^2 S\{e^{y(t)}\}$
$\frac{S\{y(t)\}}{V^{2\alpha}} - \frac{0}{V^{2\alpha}} - \frac{\pi}{V^\alpha} = \pi^2 S\{e^{y(t)}\}$
$S\{y(t)\} = V^{2\alpha}\left\{\frac{\pi}{V^\alpha} + \pi^2 S\{e^{y(t)}\}\right\}$
$S\{y(t)\} = V^{2\alpha}\frac{\pi}{V^\alpha} + V^{2\alpha}\pi^2 S\{e^{y(t)}\}$
$S\{y(t)\} = V^\alpha \pi + \pi^2 V^{2\alpha} S\{e^{y(t)}\}$ 48
Takin Sawi inverse of equation (48)
$y_0(0) = 0$, considering $y_1(t)$ from the source term and the prescribed initial conditions
$y(t) = \pi S^{-1}\{V^\alpha\} + \pi^2 S^{-1}\left\{V^{2\alpha} S\{e^{y(t)}\}\right\}$ 49
$y_1(t) = \pi S^{-1}\{V^\alpha\}$
$y_1(t) = \frac{\pi t^\alpha}{\Gamma(\alpha+1)}$ 50
$y_{n+1}(t) = \pi^2 S^{-1}\{V^{2\alpha} S\{C_n\}\}$ 51
The nonlinear term here is $L = e^{y(t)}$
From equation (9), we have
$C_0 = e^{y_0(t)})$
$C_1 = y_1(t) e^{y_0(t)}$
$C_2 = y_2(t) e^{y_0(t)} + \frac{y_1^2}{2!}(t) e^{y_0(t)}$ 52
Putting $n = 0$ into equation (48) yields
$y_2(t) = \pi^2 S^{-1}\{V^{2\alpha} S\{C_0\}\}$ 53
Substituting equation (52) into equation (53)
$y_2(t) = \pi^2 S^{-1}\left\{V^{2\alpha} S\{e^{y_0(t)}\}\right\}$
$= \pi^2 S^{-1}\{V^{2\alpha} S\{e^0\}\}$
$= \pi^2 S^{-1}\{V^{2\alpha} S\{1\}\}$
$= \pi^2 S^{-1}\{V^{2\alpha} V^{-1}\}$
$= \pi^2 S^{-1}\{V^{2\alpha-1}\}$
$y_2(t) = \frac{\pi^2 t^{2\alpha}}{\Gamma(2\alpha+1)}$ 54
Putting $n = 1$ into (51) yields
$y_3(t) = \pi^2 S^{-1}\{V^{2\alpha} S\{C_1\}\}$ 55
$= \pi^2 S^{-1}\left\{V^{2\alpha} S\{y_1(t) e^{y_0(t)}\}\right\}$
$= \pi^2 S^{-1}\left\{V^\alpha S\left\{\frac{\pi t^\alpha}{\Gamma(\alpha+1)} e^0\right\}\right\}$
$y_3(t) = \pi^2 S^{-1}\left\{V^{2\alpha} S\left\{\frac{\pi t^\alpha}{\Gamma(\alpha+1)}\right\}\right\}$
$= \pi^2 S^{-1}\left\{V^{2\alpha}\left\{\frac{\pi V^{\alpha-1}}{\Gamma(\alpha+1)}\Gamma(\alpha+1)\right\}\right\}$
$= \pi^2 S^{-1}\{\pi V^{3\alpha-1}\}$
$y_3(t) = \frac{\pi^3 t^{3\alpha}}{\Gamma(3\alpha+1)}$ 56
Putting $n = 2$ into equation (51) yields
$y_4(t) = \pi^2 S^{-1}\{V^{2\alpha} S\{C_2\}\}$ 57
$y_4(t) = \pi^2 S^{-1}\left\{V^{2\alpha} S\left\{y_2(t) e^{y_0(t)} + \frac{y_1^2}{2!}(t) e^{y_0(t)}\right\}\right\}$
$= \pi^2 S^{-1}\left\{V^{2\alpha} S\left\{\frac{\pi^2 t^{2\alpha}}{\Gamma(2\alpha+1)} e^0 + \frac{1}{2!}\left(\frac{\pi t^\alpha}{\Gamma(\alpha+1)}\right)^2 e^0\right\}\right\}$
$= \pi^2 S^{-1}\left\{V^{2\alpha} S\left\{\frac{\pi^2 t^{2\alpha}}{\Gamma(2\alpha+1)}.1 + \frac{\pi^2 t^{2\alpha}}{2!\left(\Gamma(\alpha+1)\right)^2}.1\right\}\right\}$
$= \pi^2 S^{-1}\left\{\pi^2 V^{2\alpha} S\left\{\frac{\left[2\left(\Gamma(\alpha+1)\right)^2 + \Gamma(2\alpha+1)\right]}{2\Gamma(2\alpha+1)\left(\Gamma(\alpha+1)\right)^2} t^{2\alpha}\right\}\right\}$
$= \pi^2 S^{-1}\left\{\pi^2 \frac{\left[2\left(\Gamma(\alpha+1)\right)^2 + \Gamma(2\alpha+1)\right]}{2\Gamma(2\alpha+1)\left(\Gamma(\alpha+1)\right)^2} V^{2\alpha} S\{t^{2\alpha}\}\right\}$
$= \pi^2 S^{-1}\left\{\pi^2 \frac{\left[2\left(\Gamma(\alpha+1)\right)^2 + \Gamma(2\alpha+1)\right]}{2\Gamma(2\alpha+1)\left(\Gamma(\alpha+1)\right)^2} V^{2\alpha}.\frac{V^{2\alpha-1}}{\Gamma(2\alpha+1)}\right\}$

$$= \pi^2 \left\{ \pi^2 \frac{\left[2(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{2(\Gamma(\alpha+1))^2} S^{-1}.\{V^{3\alpha-1}\} \right\}$$

$$= \pi^4 \frac{\left[2(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{2(\Gamma(\alpha+1))^2} \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$y_4(t) = \frac{\left[2(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]}{2(\Gamma(\alpha+1))^2} \cdot \frac{\pi^4 t^{3\alpha}}{\Gamma(3\alpha+1)} \qquad 58$$

The solution obtained can be written in the form

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \cdots$$

$$y(t) = 0 + \frac{\pi t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\pi^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\pi^3 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\left[2(\Gamma(\alpha+1))^2+\Gamma(2\alpha+1)\right]\pi^4 t^{3\alpha}}{2(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} \qquad 59$$

Equation (59) is the solution of equation (45) which is equal to the result obtained in Manjare and H. T. Dinde, 2020. when $\alpha = 1$, the equation(59) becomes

$$y(t) = \pi t + \frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!} + \frac{2(\pi t)^4}{4!} t^6 + \cdots$$

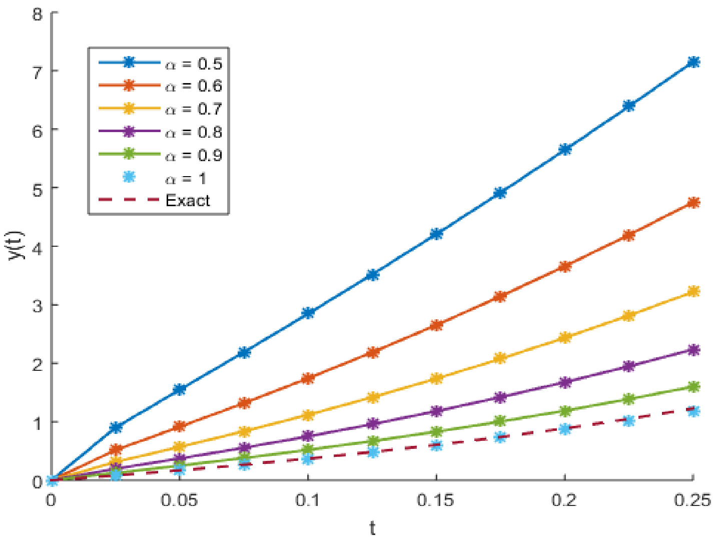

Figure 3: Behaviour of the equation when α=1.5, α=1.6, α=1.7, α=1.8, α=1.8, α=1.9, α=2.0, and exact solution

Table 3. Behaviour of the equation when $\alpha = 1.5, \alpha = 1.6, \alpha = 1.7, \alpha = 1.8, \alpha = 1.8, \alpha = 1.9, \alpha = 2.0$, and exact solution

T	$\alpha = 1.5$	$\alpha = 1.6$	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 2$	Exact Solution
0	0	0	0	0	0	0	0
0.025	0.899438	0.515683	0.312923	0.196706	0.12614	0.081705	0.081708
0.05	1.546922	0.912871	0.570659	0.372091	0.249448	0.170063	0.170118
0.075	2.19011	1.318003	0.83673	0.554887	0.379835	0.265558	0.265849
0.1	2.845546	1.741491	1.118243	0.749407	0.519363	0.368675	0.36964
0.125	3.517824	2.18645	1.417855	0.957626	0.669209	0.479899	0.482373
0.15	4.208412	2.653981	1.73688	1.180751	0.830239	0.599713	0.605119
0.175	4.917661	3.144433	2.076069	1.419649	1.003177	0.728603	0.739188
0.2	5.645464	3.657827	2.435897	1.675001	1.188666	0.867052	0.886211
0.225	6.391515	4.19403	2.816685	1.947378	1.387295	1.015546	1.048246
0.25	7.155425	4.752837	3.218667	2.237281	1.599622	1.174569	1.227947

Conclusion

In this study, we applied the Sawi Decomposition Method (SDM) to solve three fractional Bratu-type differential equations, obtaining rapidly convergent series solutions. MATLAB analysis confirmed the accuracy and stability of these solutions, demonstrating that the SDM effectively handles nonlinearities and fractional derivatives with minimal computational cost. The method's rapid convergence ensures efficiency, making it suitable for complex or large-scale problems. The findings suggest that SDM is a robust and versatile tool for solving a wide range of fractional differential equations, with potential for further application and development in various fields.

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